Esercitazioni di Analisi Matematica 2 - 12 Marzo 2025 - docente: Callegari

Exe 2: Integrale di Riemann (II)

CALCOLARE:

1 $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^4+x)\sin x \, dx$

2 $\displaystyle\int_1^2 x^3 \ln(x^4+x^2+1)\, dx$

3 $\displaystyle\int_0^1 (x^7+x^3)\arctan x^2 \, dx$

4 $\displaystyle\int_0^1 \sqrt{4-x^2}\, dx$

5 $\displaystyle\int_{\frac{13}{12}}^{\frac{5}{4}} \sqrt{x^2-1}\, dx$

$$\boxed{\textbf{6} \quad \int_{\frac{5}{12}}^{\frac{3}{4}} \frac{1}{\sqrt{x^2+1}}\, dx}$$

7 $\displaystyle\int_{-1}^{1} \frac{x+1}{(x^2+1)^2}\, dx$

8 $\displaystyle\int_4^9 \frac{1}{x+\sqrt{x}}\, dx$

NEI SEGUENTI CASI TROVARE L'ORDINE DI INFINITESIMO DI $F(x)$ PER $x \to 0^+$:

$$\boxed{\textbf{9} \quad F(x) = \int_0^x \sin(t^2)\, dt}$$

10 $F(x) = \displaystyle\int_0^{x^5} \sin(t^2)\, dt$

11 $F(x) = \displaystyle\int_{x^5}^{x^5+x^6} \sin(t^2)\, dt$

12 $F(x) = \displaystyle\int_{x^5}^{x^3+x^4} \sin(t^2)\, dt$

13 $F(x) = \displaystyle\int_0^x \int_0^t \sin(s^2)\, ds\, dt$

14 $F(x) = \displaystyle\int_0^x \sin\frac{1}{t}\, dt$

$$\lim_{x\to 0^+} \boxed{\frac{\displaystyle\int_0^x g(t)\, dt}{x^\alpha}}$$

SOLUZIONI

① $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^4+x)\sin x\,dx =$

$\displaystyle = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underline{x^4\sin x}\,dx + \overline{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin x\,dx} =$

$\displaystyle = 2\int_0^{\frac{\pi}{2}} x\sin x\,dx = 2\int_0^{\frac{\pi}{2}} x\cdot(-\cos x)'\,dx =$

$\displaystyle = 2\left(\cancel{\Big[x\cdot(-\cos x)\Big]_0^{\frac{\pi}{2}}} + \int_0^{\frac{\pi}{2}} \cos x\,dx\right) =$

0

$\displaystyle = 2\Big[\sin x\Big]_0^{\frac{\pi}{2}} = 2$

$\boxed{2}$ $\displaystyle\int_1^2 \underline{x^5}\ln(x^4+x^2+1)\,dx = \int_1^2 \left(\frac{x^6}{6}\right)'\cdot\ln(x^4+x^2+1)\,dx =$

$\displaystyle = \left[\frac{x^6}{6}\cdot\ln(x^4+x^2+1)\right]_1^2 - \int_1^2 \frac{x^6}{6}\cdot\frac{4x^3+2x}{\underline{x^4+x^2+1}}\,dx =$

$$= \int_1^2 \left(\frac{x^6}{6} - \frac{1}{6} \right)' \ln(x^4+x^2+1)\, dx =$$

$$(x^2-1)\cancel{(x^4+x^2+1)}$$

$$= \left[\left(\frac{x^6-1}{6} \right) \cdot \ln(x^4+x^2+1) \right]_1^2 - \int_1^2 \frac{x^6-1}{6} \cdot \frac{4x^3+2x}{\cancel{x^4+x^2+1}}\, dx$$

$$= \frac{63}{6} \ln(21) - \frac{1}{3} \int_1^2 (x^2-1)(2x^3+x)\, dx = [\ldots]$$

$$③ \int_0^1 (x^7+x^3) \arctan x^2 \, dx = \int_0^1 \left(\frac{x^8}{8} + \frac{x^4}{4} \right)' \arctan(x^2)\, dx =$$

$$= \left[\frac{x^8+2x^4}{8} \cdot \arctan(x^2) \right]_0^1 - \int_0^1 \frac{x^8+2x^4}{8} \cdot \frac{2x}{1+x^4} \cdot dx$$

$$= \int_0^1 \left(\frac{x^8}{8} + \frac{x^4}{4} + \frac{1}{8} \right)' \arctan(x^2)\, dx =$$

$$(x^4+1)^2$$

$$= \left[\frac{x^8+2x^4+1}{8} \cdot \arctan(x^2) \right]_0^1 - \int_0^1 \frac{x^8+2x^4+1}{\cancel{8}\,4} \cdot \frac{\cancel{2}x}{\cancel{1+x^4}}\, dx$$

$$= \frac{\pi}{8} - \frac{1}{4}\int_0^1 (x^5 + x)\, dx =$$

$$= \frac{\pi}{8} - \frac{1}{4}\left[\frac{x^6}{6} + \frac{x^2}{2}\right]_0^1 = \frac{\pi}{8} - \frac{1}{6}$$

$\boxed{4}$ $$\int_0^1 \sqrt{4-x^2}\, dx \overset{x = 2\sin t}{=} \int_0^{\frac{\pi}{6}} \sqrt{4 - 4\sin^2 t}\; 2\cos t\, dt = \ldots$$

$x = \sin t$

$$\int_0^1 \sqrt{1-x^2}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\,\cos t\, dt = \int_0^{\frac{\pi}{2}} \cos^2 t$$

$2\sqrt{\cos^2 t}$

$$\cos^2 t = \quad = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(1 + \frac{1}{2}\, \boxed{2\cos(2t)}\right) dt$$

$$\frac{1}{2}\left[t + \frac{1}{2}\sin(2t)\right]_0^{\frac{\pi}{2}} =$$

$$\boxed{\cos 2t} = \cos^2 t - \sin^2 t = \boxed{2\cos^2 x - 1}$$

$$\boxed{\cos^2 x = \frac{1 + \cos 2t}{2}}$$

. . . .

$$\operatorname{sh} t = \frac{e^t - e^{-t}}{2}$$

$$\operatorname{ch} t = \frac{e^t + e^{-t}}{2}$$

$$1 = (\cos x)^2 + (\sin x)^2$$

$$(\operatorname{sh} t)' = \left(\frac{e^t - e^{-t}}{2}\right)' = \frac{e^t + e^{-t}}{2} = \operatorname{ch} t$$

$$(\operatorname{ch} t)' = \left(\frac{e^t + e^{-t}}{2}\right)' = \frac{e^t - e^{-t}}{2} = \operatorname{sh} t$$

$$(\operatorname{ch} x)^2 - (\operatorname{sh} x)^2 \overset{?}{=} 1$$

$$\|$$

$$(\operatorname{ch} x - \operatorname{sh} x)(\operatorname{ch} x + \operatorname{sh} x)$$

$$\left(\frac{e^t + e^{-t}}{2} - \frac{e^t - e^{-t}}{2}\right)\left(\frac{e^t + e^{-t}}{2} + \frac{e^t - e^{-t}}{2}\right) =$$

$$\frac{2e^{-t}}{2} \cdot \frac{2e^t}{2} = 1$$

$$2 \operatorname{sh} t \operatorname{ch} t = 2\, \frac{e^t - e^{-t}}{2} \cdot \frac{e^t + e^{-t}}{2} =$$

$$= \frac{e^{2t} - e^{-2t}}{2} = \boxed{sh(2t)}$$

$$x = ch\, t$$

5 $\displaystyle\int_{\frac{13}{12}}^{\frac{5}{4}} \sqrt{x^2-1}\, dx \overset{\downarrow}{=}$

$$(ch\, t)^2 - (sh\, t)^2 = 1$$

$$\boxed{(ch\, t)^2 - 1 = (sh\, t)^2}$$

$$(ch\, t)^2 = (sh\, t)^2 + 1$$

$$= \int \sqrt{ch^2 t - 1} \cdot sh\, t\, dt$$

$$\frac{13}{12} = ch(t_0)$$

$$\frac{e^{t_0} + e^{-t_0}}{2} = \boxed{\frac{13}{12}} = \frac{5}{4}$$

$$e^{t_0} + \frac{1}{e^{t_0}} = \boxed{\frac{13}{6}} \qquad (e^{t_0})^2 - \frac{13}{6} e^{t_0} + 1 = 0$$

$$\lambda^2 - \frac{13}{6}\lambda + 1 = 0 \qquad (\lambda - 2)(2\lambda - 1)$$

$$6\lambda^2 - 13\lambda + 6 = 0 \qquad 2\lambda^2 - 5\lambda + 2 = 0$$

$$6\lambda^2 - 9\lambda - 4\lambda + 6 = 0 \qquad 2\lambda^2 - 4\lambda - \lambda + 2 = 0$$

$$2\lambda(\lambda - 2) - (\lambda - 2) = 0$$

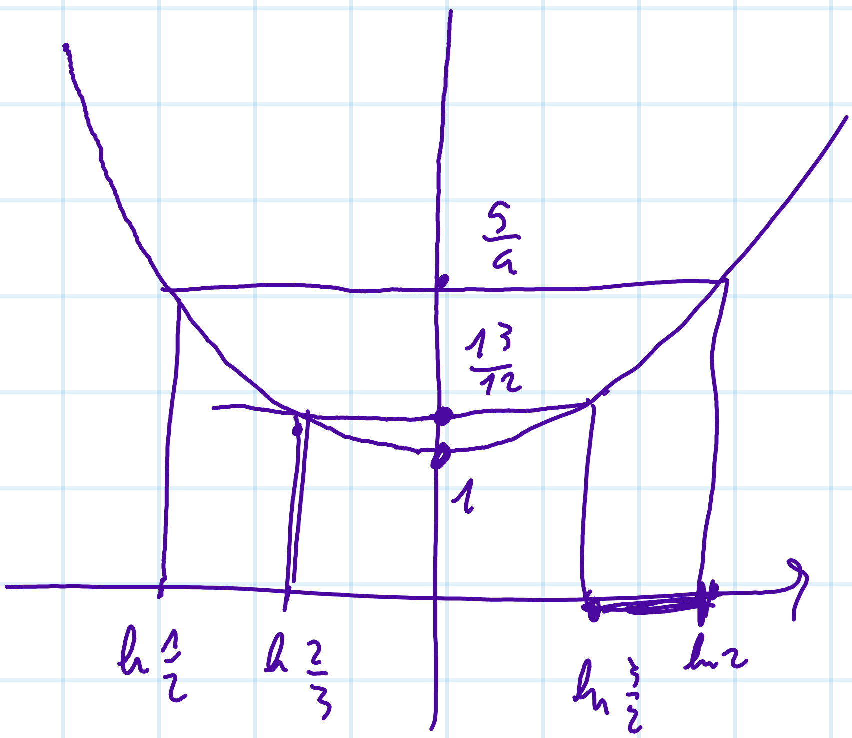
$$3\lambda(2\lambda-3)-2(2\lambda-3)=0$$

$$(2\lambda-3)(3\lambda-2)=0$$

$$\lambda=\frac{3}{2} \qquad \lambda=\frac{2}{3}$$

$$e^t=\frac{3}{2} \qquad e^t=\frac{2}{3}$$

$$t=\ln\left(\frac{3}{2}\right. \qquad t=\ln\frac{2}{3}$$



$$\boxed{\ln\frac{3}{2}\le t\le \ln 2}$$

$\lambda=e^t$

su $[\ln\frac{3}{2}, \ln 2]$

$$\dots = \int_{\ln\frac{3}{2}}^{\ln 2} \sqrt{(\operatorname{ch} t)^2 - 1} \cdot \operatorname{sh} t \, dt =$$

$$= \sqrt{(\operatorname{sh} t)^2} \leftarrow |\operatorname{sh} t|$$

$$\uparrow$$

$$\operatorname{sh} t$$

$$= \int_{\ln\frac{3}{2}}^{\ln 2} \operatorname{sh} t \cdot \operatorname{sh} t \, dt =$$

$$= \int_{\ln\frac{3}{2}}^{\ln 2} \left(\frac{e^{t} - e^{-t}}{2}\right)^2 dt = \qquad \int$$

$$\operatorname{sh}^2 t$$

$$\frac{e^{2t} + e^{-2t} - 2}{4} = \frac{1}{2}\left(\frac{e^{2t} + e^{-2t}}{2} - 1\right)$$

$$= \frac{1}{2}\left(\operatorname{ch}(2t) - 1\right)$$

$$= \frac{1}{4}\int_{\ln\frac{3}{2}}^{\ln 2} e^{2t} + e^{-2t} - 2 \, dt =$$

$$= \frac{1}{4}\left[\frac{1}{2}e^{2t} - \frac{1}{2}e^{-2t} - 2t\right]_{\ln\frac{3}{2}}^{\ln 2} =$$

$$= \frac{1}{4}\left(\left(\frac{1}{2}e^{2\ln 2} - \frac{1}{2}e^{-2\ln 2} - 2\ln 2\right)\right.$$

$$\left. - \frac{1}{2}e^{2\ln\frac{3}{2}} + \frac{1}{2}e^{-2\ln\frac{3}{2}} + 2\ln\frac{3}{2}\right)$$

$$\frac{1}{4}\left(2 - \frac{1}{8} - 2\ln 2 - \frac{9}{8} + \frac{2}{9} + 2\left(\ln\frac{3}{2}\right)\right)$$

$$2\left(\ln\frac{3}{2} - \ln 2\right) = 2\ln\frac{3}{4}$$

$$= \frac{1}{4}\left(2\ln\frac{3}{4} - \frac{10}{8} + 2 + \frac{2}{9}\right) = \ldots$$

$$\lim_{x\to 0^+} \frac{\int_0^x \sin(t^2)\,dt}{x^\alpha} \overset{H}{=} \lim \frac{\sin(x^2)}{\alpha x^{\alpha-1}} =$$

$$\lim_{x\to 0^+} \frac{\int_0^x \sin\frac{1}{t}\,dt}{x^\alpha} \overset{H}{=} \lim_{x\to 0^+} \frac{\sin\frac{1}{x}}{\alpha x^{\alpha-1}}$$

$$-1 \le \sin\frac{1}{t} \le 1$$

$$\frac{\int_0^x (-1)\,dt}{x^{\frac12}} \le \frac{\int_0^x \sin\frac{1}{t}\,dt}{x^{\frac12}} \le \frac{\int_0^x 1\,dt}{x^{\frac12}}$$

$$-\frac{x}{\sqrt{x}} \le \quad \le \frac{x}{\sqrt{x}}$$

$$-\sqrt{x} \le \quad \le \sqrt{x}$$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \qquad\qquad 0$$

$\int_0^x =$

$\frac{1}{n\pi}$

$\int_0^x - \int_0^{\bar{x}_n}$

$F(x) - F(\bar{x}_n)$

$F(x) = \int_0^x \sin\frac{1}{t}\,dt$

$\int_{\bar{x}}^{x} \leq x - \bar{x}_n$

$f(x) \leq$

$F\left(\frac{1}{2k\pi}\right) \geq 0$

$\frac{1}{(n+1)\pi} - \frac{1}{n\pi}$

$\boxed{\frac{1}{\pi(n+1)} < x < \frac{1}{n\pi}}$

$F\left(\frac{1}{(2k+1)\pi}\right) < 0$

$\boxed{\frac{1}{(n+1)n\pi}}$

$\left(\frac{1}{n+1}\right)^2 < (\pi x)^2$

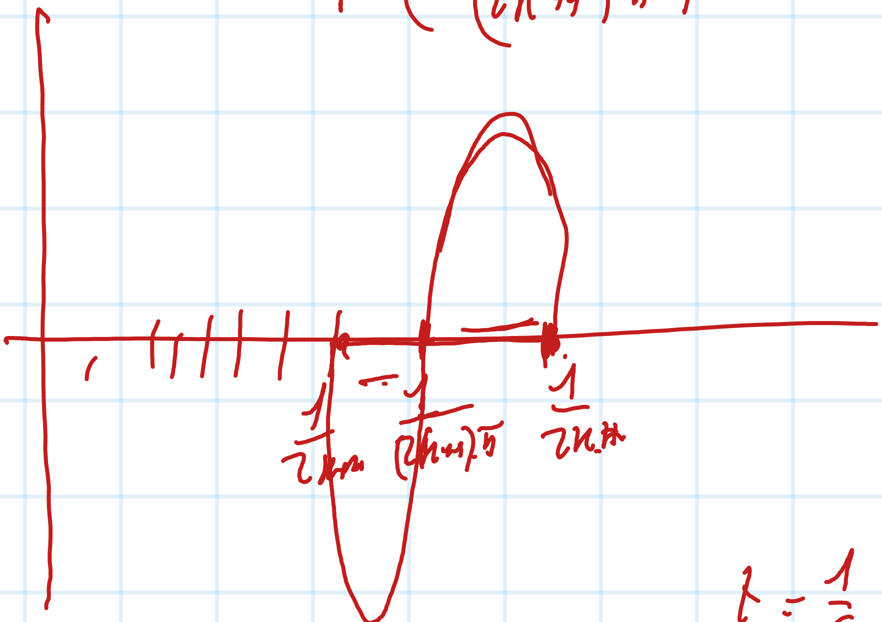
$\frac{1}{\pi n(n+1)} = \frac{1}{\pi}\left(\frac{n+1}{n}\right)\frac{1}{(n+1)^2} \quad n \left(\frac{1}{n+1}\right) = \frac{1}{\pi}\frac{1}{(n+1)^2}\frac{n+1}{n}$

$\leq \frac{1}{2\pi}\frac{1}{(n+1)^2}$

$$F\left(\frac{1}{2n\pi}\right) > 0$$

$$F\left(\frac{1}{(2n+1)\pi}\right) < 0$$

$$\frac{1}{t} = n\pi$$



$$t = \frac{1}{s}$$

$$\frac{1}{t} = s$$

$$\int_{\frac{1}{(2n+2)\pi}}^{\frac{1}{2n\pi}} \sin\frac{1}{t}\, dt = \int_{(2n+2)\pi}^{2n\pi} \sin s \cdot \left(-\frac{1}{s^2}\right) ds =$$

$$= \int_{2n\pi}^{(2n+2)\pi} \frac{\sin s}{s^2}\, ds = \left[\int_{2n\pi}^{(2n+1)\pi} \frac{\sin s}{s^2} + \int_{(2n+1)\pi}^{(2n+2)\pi} \frac{\sin s}{s^2}\right.$$

$=$

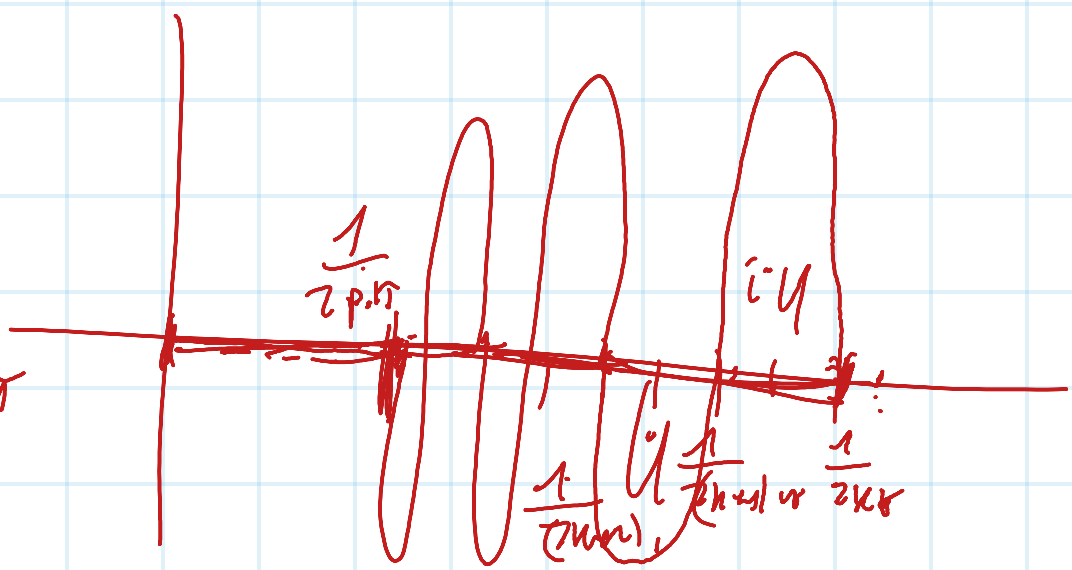
$$\int_{(2k+1)\pi}^{(2k+2)\pi} \frac{\sin s}{s^2}\,dx = \int_{2k\pi}^{(2k+1)\pi} \frac{\boxed{\sin(u+\pi)}}{(u+\pi)^2}\,du \qquad s = u+\pi$$

$-\sin u$

$$\boxed{\int_{2k\pi}^{(2k+1)\pi} \frac{-\sin s}{(s+\pi)^2}\,ds}$$

$$= \int_{2k\pi}^{(2k+1)\pi} \sin s \left(\frac{1}{s^2} - \frac{1}{(s+\pi)^2} \right) ds > 0$$

$$\int_0^{\frac{1}{2k\pi}}$$



$\forall p, k \qquad p > k$

$$F\left(\frac{1}{2k\pi}\right) \int_0^{\frac{1}{2k\pi}} \sin \frac{1}{t}\,dt.$$

$$\int_{\frac{1}{2p\pi}}^{\frac{1}{2k\pi}} \sin \frac{1}{t}\,dt > 0$$